Problem Sheet 12

Exercise 12.1

Let $f_n, f : \mathcal{X} \to \mathcal{Y}$ be measurable maps on Banach spaces and such that $f_n(x_n) \to f(x)$ for any sequence x_n in $\mathcal{X}$ converging to x. Prove that f is continuous and f_n converges to f locally uniformly.

Exercise 12.2

Prove the following Continuous Mapping Theorem:

Let $f_n, f : \mathcal{X} \to \mathcal{Y}$ be measurable maps between metric spaces such that $f_n(x_n) \to f(x)$ for any sequence x_n in $\mathcal{X}$ converging to x. If $\mu_n, \mu \in \mathbb{P}(\mathcal{X})$ with $\mu_n \to \mu$, Then $(f_n)_*\mu_n \to f_*\mu$. In particular, if ξ_n are random variables converging to ξ in distribution, then $f_n(\xi_n)$ converges to $f(\xi)$ in distribution.

Exercise 12.3

Let P be a transition function on $\mathcal{X}$ and let $V : \mathcal{X} \to \mathbb{R}_+ \cup \{\infty\}$ be a Borel measurable function. Suppose there exist a positive constant $\gamma \in (0, 1)$ and a constant $C > 0$ such that

$$TV(x) \leq \gamma V(x) + C \ ,$$

for every x such that $V(x) \neq \infty$. Then

$$T^n V(x) \leq \gamma^n V(x) + \frac{C}{1-\gamma}.$$

Exercise 12.4

Let $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be continuous and bounded, (ξ_n) a collection of i.i.d $\mathcal{Y}$-valued random variables independent of the $\mathcal{X}$-valued random variable X_0. Define the Markov Process

$$X_{n+1} = F(X_n, \xi_n).$$

Suppose, furthermore, there exists a Borel measurable function $V : \mathcal{X} \to \mathbb{R}_+$ with compact sub-level sets and constants $\gamma \in (0, 1)$ and $C \geq 0$ such that

$$\int_{\mathcal{Y}} V\big(F(x, y)\big)\, \hat{\mathbb{P}}(dy) \leq \gamma V(x) + C \ , \quad \forall x \in \mathcal{X} \ ,$$

where $\hat{\mathbb{P}}$ is the distribution of ξ_n. Prove that the process X has at least one invariant probability measure.